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Generalized pulsating strings

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Abstract: In this paper we consider new solutions for pulsating strings. For this purpose we use the idea of the generalized ansatz for folded and circular strings in [1]. We find the solutions to the resulting Neumann-Rosochatius integralbe system and the corrections to the energy. To do that we use the approach developed by Minahan in [2] and find that the corrections are quite different from those obtained in that paper and in [3]. We conclude with comments on our solutions and the obtained corrections to the energy, expanded to the leading order in lambda.

Keywords: Superstrings and Heterotic Strings, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

The recent developments in string theory is basically related to searching more and more examples supporting and verifying AdS/CFT correspondence. After a huge amount of papers checking the conjecture at supergravity level, a new approaches treating the problem beyond the SUGRA limit appeared. First of all, the significan development in our understanding of string theory in $AdS_5 \times S^5$ background was inspired by the idea of Berenstein, Maldacena and Nastase (BMN) to analyse the string theory in the visinity of null geodesics of $AdS_5 \times S^5$ space [4]. Expanding string sigma model around the geodesics one ends up with so called pp-wave geometry and procedure tunrs out to be so called Penrose limit. The remarkable properties of this background are that it has a maximal amount of supersymmetry and the string sigma model in this background turns out to be exactly solvable. Putting all these together, one can conclude that the resulting theory is semiclassical string theory of point-like string moving along the geodesics and having large angular momentum J.

The picture emerging from BMN limit was understood by using another idea - semiclassical analysis of string theory in $AdS_5 \times S^5$ background [5]. The key point in this approach is that large classical charges means small σ-model corrections so that the semiclassical results are reliable enough to compute the dimensions of the operators from SYM side. This approach enables us to consider more generic operators than just BPS ones. In a considerable amount of papers [1], [6]–[16] and [18]–[22][1] various string configurations were analysed and the anomalous dimensions of the corresponding SYM operators were obtained. One should note that most of the examples of string configurations corresponding to generically non-BPS operators share one property - in the large energy limit all they become supersymmetric [23, 24]. One of the exeptions from the so called nearly"BPS strings are configurations named pulsating strings [2, 3]. The corresponding operators also have large charges, but are not necessarily close to any BPS one.

[1] For more complete list see the references in these papers.

In this paper we are looking for more general solutiuons belonging to the class of pulsating strings. Before we go into details, let us remind some of the general properties of the pulsating strings. Let us consider pulsating string in S^5 part of $AdS_5 \times S^5$ geometry. The string states then should be characterized by the charges associated to the isometry group of S^5, SO(6). The highest weight representation (J_1, J_2, J_3) is actually defined by these angular momenta. On gauge theory side we know that SYM theory can be characterized by three R-charges (J_1, J_2, J_3) which define the highes weight representation of SO(6) gauge group with Dynkin indices $[J_2+J_3, J_1-J_2, J_2-J_3]$ [8]. The operators in SYM corresponding to the above R-charges are of the form

$$\mathrm{Tr}\left(X^{J_1} Y^{J_2} Z^{J_3}\right), \tag{1.1}$$

where X, Y and Z are three complex scalars in the adjoint of SO(6) forming $N = 4$ supermultiplet. Due to the mixing between the operators, the computation of the anomalous dimensions becomes a non-trivial problem even at one loop. Going back, one should note that for different string configurations known in the literature there are more ot less complicated relations between the anomalous dimensions and the R-charges. As we already mentioned above, the pulsating strings are one of the examples which has more special status. The goal of this paper is to generalize the known solutions for pulsating strings [2, 3] and to find the corrections to the classical energy, which means to obtain the anomalous dimensions. The paper is organized as follows. In the next section 2 we briefly review the known in the literature solutions. In section 3, based on the generalized string ansatz for folded and circular string of [1] we find new solutions for pulsating strings. In the section 4 we find the corrections to the classical energy for our solutions and find the anomalous dimension of the corresponding SYM operators.

2. Review of pulsating strings

In this section we give a brief review of the pulsating string solutions obtained first by Minahan [2] and generalized later in [3]. We will concentrate only on the case of pulsating string on S^5 part of $AdS_5 \times S^5$, i.e. we consider a circular string which pulsates expanding and contracting on S^5. To fix the notations, we start with the metric of S^5 and relevant part of AdS_5

$$ds^2 = R^2\left(\cos^2\theta d\Omega_3^2 + d\theta^2 + \sin^2\theta d\psi^2 + d\rho^2 - cosh^2 dt^2\right), \tag{2.1}$$

where $R^2 = 2\pi\alpha'\sqrt{\lambda}$. To obtain the simplest solution, we identify the target space time with the worldsheet one, $t = \tau$, and use the ansatz $\psi = m\sigma$, i.e. the string is stretched along ψ direction, $\theta = \theta(\tau)$ and keep the dependence on $\rho = \rho(\tau)$ for a while. The reduced Nambu-Goto action in this case is

$$S = m\sqrt{\lambda}\int dt \sin\theta\sqrt{cosh^2\rho - \dot{\theta}^2}. \tag{2.2}$$

For our considerations it is useful to pass to hamiltonian formulation. For this purpose we find the canonical momenta and find the hamiltonian in the form[2]

$$H = cosh\rho\sqrt{\Pi_\rho^2 + \Pi_\theta^2 + m^2\lambda\sin^2\theta}. \tag{2.3}$$

Fixing the string to be at the origin of AdS_5 space ($\rho = 0$), we see that the squared hamiltonian have a form very similar to a point particle. The last term in (2.3) can be considered as a perturbation, so first we find the wave function for a free particle in the above geometry

$$\frac{\cosh\rho}{\sinh^3\rho}\frac{d}{d\rho}\cosh\rho\sinh^3\rho\frac{d}{d\rho}\Psi(\rho,\theta) - \frac{\cosh^2\rho}{\sin\theta\cos^3\theta}\frac{d}{d\theta}\sin\theta\cos^3\theta\frac{d}{d\theta}\Psi(\rho,\theta) = E^2\Psi(\rho,\theta)\,. \tag{2.4}$$

The solution to the above equation is

$$\Psi_{2n}(\rho,\theta) = (\cosh\rho)^{-2n-4}\,P_{2n}(\cos\theta)\,, \tag{2.5}$$

where $P_{2n}(\cos\theta)$ are spherical harmonics on S^5 and the energy is given by

$$E_{2n} = \Delta = 2n + 4\,. \tag{2.6}$$

Since we consider highly excited states (large energies), one should take large n, so we can approximate the spherical harmonics as

$$P_{2n}(\cos\theta) \approx \sqrt{\frac{4}{\pi}}\cos(2n\theta)\,. \tag{2.7}$$

The correction to the energy can be obtained by using perturbation theory, which to first order is

$$\delta E^2 = \int\limits_0^{\pi/2} d\theta\,\Psi_{2n}^\star(0,\theta)\,m^2\lambda\sin^2\theta\,\Psi_{2n}(0,\theta) = \frac{m^2\lambda}{2}. \tag{2.8}$$

Up to first order in λ we find for the anomalous dimension of the corresponding YM operators[3]

$$\Delta - 4 = 2n\left[1 + \frac{1}{2}\frac{m^2\lambda}{(2n)^2}\right]. \tag{2.9}$$

On should note that in this case the R-charge is zero. In order to include it, we consider pulsating string on S^5 which has a center of mass that is moving on the S^3 subspace of S^5 [3]. While in the previous example S^3 part of the metric was assumed trivial, now we consider all the S^3 angles to depend on τ (only). The corresponding Nambu-Goto action now is

$$S = -m\sqrt{\lambda}\int dt\sin\theta\sqrt{1 - \dot\theta^2 - \cos^2\theta g_{ij}\dot\phi^i\dot\phi^j}\,, \tag{2.10}$$

[2]For more details see [2].

[3]See [2] for more details.

where ϕ_i are S^3 angles and g_{ij} is the corresponding S^3 metric. The hamiltonian in this case is [3]

$$H = \sqrt{\Pi_\theta^2 + \frac{g^{ij}\Pi_i\Pi_j}{\cos^2\theta} + m^2\lambda\sin^2\theta}\,. \tag{2.11}$$

Again, we see that the squared hamiltonian looks like the point particle one, however, now the potential has angular dependence. Denoting the quantum number of S^3 and S^5 by J and L correspondingly, one can write the Schrodinger equation

$$-\frac{4}{\omega}\frac{d}{d\omega}\Psi(\omega) + \frac{J(J+1)}{\omega}\Psi(\omega) = L(L+4)\Psi(\omega), \tag{2.12}$$

where we set $\omega = \cos^2\theta$. The solution to the Schrodinger equation is

$$\Psi(\omega) = \frac{\sqrt{2(l+1)}}{(l-j)!}\frac{1}{\omega}\left(\frac{d}{d\omega}\right)^{l-j}\omega^{l+j}(1-\omega)^{l-j}, \quad j = \frac{J}{2}; l = \frac{L}{2}\,. \tag{2.13}$$

The first order correction to the energy δE is

$$\delta E^2 = m^2\lambda\,\frac{2(l+1)^2 - (j+1)^2 - j^2}{(2l+1)(2l+3)}\,, \tag{2.14}$$

or, up to first order in λ

$$E^2 = L(L+4) + m^2\lambda\frac{L^2 - J^2}{2L^2}\,. \tag{2.15}$$

The anomalous dimension then is given by

$$\gamma = \frac{m^2\lambda}{4L}\alpha(2-\alpha)\,, \tag{2.16}$$

where $\alpha = 1 - J/L$.

We conclude this section reffering for more details to [2] and [3].

3. New pulsating string solutions

In this section we consider in details generalized solutions generated by the more general ansatz suggested in [1]. After a short review of the generalized ansatz we will analyse in some details the solution of the resulting dynamical system.

First of all, we will consider the puslating string solutions in S^5 part of $AdS_5 \times S^5$ geometry of the target spacetime. To start with, let us mention the generalized ansatz for the string embedding coordinates:

$$\begin{aligned} t &= \kappa\tau \\ X_i &= r_i(\tau)e^{i\alpha(\tau)+im_i\sigma} = z_i(\tau)e^{im_i\sigma} \\ \sum_{i=1}^{3} r_i^2(\tau) &= 1 \end{aligned} \tag{3.1}$$

and other coordinates in AdS_5 part set to appropriate constant values. In order to ensure singlevelued parametrization of the string worldsheet the constants m_i should be integers. Next, to ensure vanishing of the non-Cartan components of $O(6)$ angular momentum, one should take $m_i \neq m_j$. No we are ready, using the ansatz (3.1), to write down the lagrangian of the constrained sigma model describing our string dynamics:

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{3}\left(\dot{z}_i\dot{\bar{z}}_i - m_i^2 z_i\bar{z}_i\right) + \frac{1}{2}\Lambda\left(\sum_{i=1}^{3} z_i\bar{z}_i - 1\right), \tag{3.2}$$

where $z_i(\tau)$ are defined in (3.1), Λ is a lagrange multiplier and the coordinates satisfy the constraints

$$z_i\bar{z}_i = r_i^2 \tag{3.3}$$

(i.e., we are dealing with constrained SO(6) sigma model). In terms of the real variables (r_i, α_i) the lagrangian can be written in the form

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{3}\left[\dot{r}_i^2 + r_i^2\dot{\alpha}_i^2 - m_i^2 r_i^2\right] + \frac{1}{2}\Lambda\left(\sum_{i=1}^{3} r_i^2 - 1\right). \tag{3.4}$$

The equations of motion for α_i can be easily solved in terms of r_i:

$$\frac{d}{d\tau}\left(r_i^2\dot{\alpha}_i\right) = 0 \quad\Rightarrow\quad \dot{\alpha}_i = \frac{\mathcal{J}_i}{r_i^2} \tag{3.5}$$

so that the lagrangian get easily reduced to the form

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{3}\left[\dot{r}_i^2 - m_i^2 r_i^2 - \frac{\mathcal{J}_i^2}{r_i^2}\right] + \frac{1}{2}\Lambda\left(\sum_{i=1}^{3} r_i^2 - 1\right), \tag{3.6}$$

and m_i and $\mathcal{J}_i$ must satisfy the costraint

$$\sum_i m_i\,\mathcal{J}_i = 0\,.$$

As in the case of folded and circular strings, one can consider two cases — of constant lagrange multiplier Λ and non-constant one.

Let us focus first on the case of constant Λ. This case is the simplest example of pulsating string solutions. If we use complex notations, the lagrangian (3.2) leads to a simple equation of motion

$$\ddot{z}_i + (m_i^2 - \Lambda)z_i = 0\,. \tag{3.8}$$

If we denote $l_i^2 = m_i - \Lambda$ and use the fact that Λ is a constant, the equation (3.8) has the following general solution

$$z_i = a_i e^{il_i\tau} + b_i e^{-il_i\tau}\,. \tag{3.9}$$

The assumption that Λ is a constant implies that one can set $b_i = 0$ and the solution then becomes

$$z_i = a_i e^{il_i\tau} \tag{3.10}$$

supplied with the constraint

$$\sum_{i=1}^{3} a_i^2 = 1\,. \tag{3.11}$$

Now we turn to the case of non-constant lagrange multiplier Λ. We will look for a solution of the constrained sigma model choosing

$$\begin{aligned} v_i &= 0 \\ \alpha_i &= 0 \\ r_3 &= 0\,. \end{aligned} \tag{3.12}$$

i.e.

$$x_i = r_i(\tau) e^{i m_i \sigma}\,, \qquad \sum_{i=1}^{2} r_i^2(\tau) = 1\,. \tag{3.13}$$

In this case the lagrangian and the lagrange multiplier become

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{2}\left(\dot{r}_i^2 - m_i^2 r_i^2\right) + \frac{1}{2}\Lambda\left(\sum_{i=1}^{2} r_i^2 - 1\right) \tag{3.14}$$

$$\Lambda = -\sum_{i=1}^{2}\left(\dot{r}_i^2 - m_i^2 r_i^2\right). \tag{3.15}$$

The equations of motion following from the above lagrangian are

$$\ddot{r}_i = -m_i^2 r_i + \Lambda r_i\,. \tag{3.16}$$

We solve the equations using the natural parametrization of S^3 in terms of the angles γ and ψ

$$\begin{aligned} r_1 &= \sin\gamma\cos\psi \\ r_2 &= \sin\gamma\sin\psi \\ r_3 &= \cos\gamma\,. \end{aligned} \tag{3.17}$$

The choice (3.12) means that we set $\gamma = \pi/2$. The resulting equation for the only dinamical variable ψ becomes

$$\ddot{\psi} + m_{21}^2 \sin\psi\cos\psi = 0\,, \tag{3.18}$$

where $m_{21}^2 = m_2^2 - m_1^2$. Assuming that $\psi(0) = 0$, we find

$$A\tau = \int_0 \frac{d\psi}{\sqrt{1 - \frac{m_{21}^2}{A^2}\sin^2\psi}}\,, \tag{3.19}$$

where A is an integration constant. In what follows we will consider the case of

$$\left(\frac{m_{21}^2}{A}\right)^2 \leq 1 \tag{3.20}$$

and the opposite case can be considered in the same way (we note that the condition (3.20) ensures the existing of turning point, i.e. the string is pulsating). The solution for ψ then is

$$\sin\psi = sn\left(A\tau, t^2\right)\,, \tag{3.21}$$

where the modulus t^2 is defined as $t^2 = m_{21}^2/A^2$. If we choose the period of pulsations to be 2π, the integration constant is determined by this condition to

$$A = \frac{2K(t)}{\pi}, \tag{3.22}$$

where $K(t)$ is the complete elliptic integral of first kind.

Now we are to give essential generalization of the pulsating string solution presented above. The case we will consider is of non-constant Λ but with

$$v_3 = 0, \quad \alpha_3 = 0\,, \qquad r_3 = 0\,. \tag{3.23}$$

Using the parametrization (3.1) and the constraints (3.23) one can find for our pulsating string the following lagrangian

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{2}\left(\dot{r}_i^2 - m_i^2 r_i^2 - \frac{\mathcal{J}_i^2}{r_i^2}\right) + \frac{1}{2}\Lambda\left(\sum_{i=1}^{2} r_i^2 - 1\right), \tag{3.24}$$

where we substituted for $\dot{\alpha} = \mathcal{J}_i^2/r_i^2$ (which follows from the equation for α). Solving for the lagrange multiplier Λ we find

$$\Lambda = -\sum_{i=1}^{2}\left(\dot{r}_i^2 - m_i^2 r_i^2 - \frac{\mathcal{J}_i^2}{r_i^2}\right). \tag{3.25}$$

The equations of motion are

$$\ddot{r}_i = -m_i^2 r_i - \frac{\mathcal{J}_i^2}{r_i^3} - r_i\left(\sum_{j=1}^{2}\dot{r}_j^2 - m_j^2 r_j^2 - \frac{\mathcal{J}_i^2}{r_j^2}\right). \tag{3.26}$$

Using the explicit parametrization (3.1) one can find the following equation for the angle ψ

$$\ddot{\psi} + m_{21}^2 \sin\psi\cos\psi + \frac{\mathcal{J}_2^2\cos\psi}{\sin^3\psi} - \frac{\mathcal{J}_1^2\sin\psi}{\cos^3\psi} = 0\,. \tag{3.27}$$

Integrating once the above equation one finds

$$\dot{\psi} = \frac{\sqrt{\mathcal{J}_2^2 + (A + \mathcal{J}_1^2 - \mathcal{J}_2^2)\sin^2\psi - (A + m_{21}^2)\sin^4\psi + m_{21}^2\sin^6\psi}}{\sin\psi\cos\psi}. \tag{3.28}$$

Defining $X = \sin^2\psi/2$ we get

$$\frac{dX}{d\tau} = \sqrt{\mathcal{J}_2^2 + aX - bX^2 + 8m_{21}^2 X^3}\,, \tag{3.29}$$

where

$$\begin{aligned} a &= 2(A + \mathcal{J}_1^2 - \mathcal{J}_2^2) \\ b &= 4(A + m_{21}^2)\,. \end{aligned} \tag{3.30}$$

Obviously this equation has a solution in terms of Jacobi elliptic functions. To bring the equation (3.29) into Jacobi canonical form we use the change of variables

$$\alpha + \xi \,. \tag{3.31}$$

If we denote by α the root of the equation

$$8m_{21}^2\alpha^3 - b\alpha^2 + a\alpha + \mathcal{J}_2^2 = 0\,, \tag{3.32}$$

one can write the equation (3.29) in the form

$$\frac{d\xi}{d\tau} = 8m_{21}^2\xi_+\xi_-\left(1 - \frac{\xi}{\xi_-}\right)\left(1 - \frac{\xi}{\xi_+}\right), \tag{3.33}$$

where $\xi_\pm$ are the solutions of the equation

$$\xi^2 - \frac{b - 24m_{21}^2\alpha}{8m_{21}^2}\xi - \frac{8m_{21}^2 + \alpha b + \alpha(b - 24m_{21}^2)}{8m_{21}^2} = 0\,.$$

Finally, defining a new variable by

$$\eta^2 = \frac{\xi}{\xi_-}\,,$$

we find the standard equation for the Jacobi elliptic functions

$$\left(\frac{d\eta}{d\tau}\right) = \frac{8m_{21}^2\xi_+}{4}\left(1 - \eta^2\right)\left(1 - t\eta^2\right), \tag{3.36}$$

where the modulus t is defined by

$$t = \frac{\xi_-}{\xi_+}\,, \tag{3.37}$$

and the solution takes the form

$$\eta = sn\left(\frac{\sqrt{8m_{21}^2\xi_+}}{2}\tau, t\right). \tag{3.38}$$

In the next section we proceed with calculation of the corrections to the energy for the system we just described.

4. Corrections to the classical energy

In this section we calculate the corrections to the classical energy using the approach developed in [2, 3]. Let us consider a circular pulsating string expanding and contracting on S_5, which has a center of mass that is moving on an S_3 subspace. We will assume that the string is with fixed spatial coordinates in AdS_5 (exept the time variable). The parametrization of the geometry is as follows:

$$\begin{aligned} X_1 + iX_2 &= \sin\gamma\cos\chi\exp(i\varphi_1) \\ X_3 + iX_5 &= \sin\gamma\cos\chi\exp(i\varphi_2) \\ X_5 + iX_6 &= \sin\gamma\cos\chi\exp(i\varphi_3)\,, \end{aligned} \tag{4.1}$$

so the relevant metric is:

$$ds^2 = R^2(-dt^2 + d\gamma^2 + \cos^2\gamma\, d\varphi_3^2 + \sin^2\gamma\, d\,\Omega_3^2)\,, \quad = R^2(-dt^2 + d\Omega_5^2)\,, \tag{4.2}$$

where $d\Omega_3^2$ and $d\Omega_5^2$ are the line elements of S^3 and S^5 respectively. The corresponding metrics are $g^{(3)}_{ab} = diag(1, \cos^2\chi, \sin^2\chi)$, $g^{(5)}_{ij} = diag(1, \cos^2\gamma, \sin^2\gamma\, g(3)_{ij})$ and $R^2 = 2\pi\alpha'\sqrt{\lambda}$. If we identify t with τ and use following classical anzatz:

$$\gamma = \gamma(\tau)\,, \quad \chi = \chi(\tau)\,, \quad \varphi^i = n^i\sigma + \alpha^i(\tau)\,, \quad i = 1, 2, 3, \tag{4.3}$$

the Nambu-Goto action

$$S = -\frac{1}{2\pi\alpha'}\int d\tau\, d\sigma \sqrt{-det(\partial_\alpha X^\mu \partial_\beta X_\mu)} \tag{4.4}$$

then reduces to

$$\begin{aligned} S\, &= -\sqrt{\lambda}\int d\tau\, d\sigma\, \times \\ &\times \sqrt{(g^{(5)}_{ij} h^i h^j)\left[1 - g^{(5)}_{ij}\dot{y}^i\dot{y}^j\right] + \left(g^{(5)}_{ij} h^i \dot{y}^j\right)^2}\,, \end{aligned} \tag{4.5}$$

where

$$\begin{array}{lllll} y_1 = \gamma(\tau)\,; & y_2 = \alpha_3(\tau)\,; & y_3 = \chi(\tau)\,; & y_4 = \alpha_1\,; & y_5 = \alpha_2(\tau)\,; \\ h_1 = 0\,; & h_2 = n_3\,; & h_3 = 0\,; & h_4 = n_1\,; & h_5 = n_2\,. \end{array} \tag{4.6}$$

Now we are going to apply the procedure for calculation of the anomalous dimensions developed in [2]. For this puspose, we find first the canonical momenta of our system. Straightforward calculations give for the momenta

$$\Pi_k = \frac{\sqrt{\lambda}\left[(g^{(5)}_{ij} h^i h^j)\,\dot{y}^l - \left(g^{(5)}_{ij} h^i \dot{y}^j\right) h^l\right] g^{(5)}_{lk}}{\sqrt{(g^{(5)}_{ij} h^i h^j)\left[1 - g^{(5)}_{ij}\dot{y}^i\dot{y}^j\right] + \left(g^{(5)}_{ij} h^i \dot{y}^j\right)^2}}\,. \tag{4.7}$$

Solving for the derivatives in terms of the canonical momenta and substituting back into the hamiltonian, we find

$$H = \sqrt{g^{(5)}_{ij}\Pi^i\Pi^j + V(\gamma,\chi)}\,, \tag{4.8}$$

or

$$H^2 = \Pi_\gamma^2 + \frac{\Pi^2_{\alpha_3}}{\cos^2\gamma} + \frac{\Pi_{\alpha_a}\Pi_{\alpha_b} g^{(3)\,ab}}{\sin^2\gamma} + V(\gamma,\chi)\,. \tag{4.9}$$

Since we consider high energies, one can think of this hamiltonian as of square root of a point particle one.[4] Hence, we can consider the potential terms as a perturbation. The potential itself has the form

$$\begin{aligned} V(\gamma,\chi)\, &= \lambda\left[\cos^2\gamma(n^3)^2 + \sin^\gamma(\cos^2\chi(n^1)^2 + \sin^2\chi(n^2)^2)\right] \\ &= \lambda\left[(n^3)^2 + \sin^2\gamma[(n^2)^2 - (n^3)^2 + \sin^2\chi((n^2)^2 - (n^3)^2)]\right]. \end{aligned} \tag{4.10}$$

[4]When we apply quasiclassical quantization we are dealing actually with a family of solutions. Each of them looks like almost as for point particle (for more comments see for instance [2, 3]).

The above perturbation to the free action will produce the corrections to the energy and therefore, to the anomalous dimension. Thus, we proceed with the consideration of the free wave-functions on S_5 and S_3, and then use the perturbation theory to first order to find the correction of order λ. The total S_5 angular momentum quantum number will be denoted by L and the total angular momentum quantum number on S_3 is J. In these notations we have for S_3 and S_5 :

$$\triangle(S_3) = \frac{1}{\sin\chi\cos\chi}\frac{\partial}{\partial\chi}\left[\sin\chi\cos\chi\frac{\partial}{\partial\chi}\right] + \frac{1}{\cos^2\chi}\frac{\partial^2}{\partial(\varphi^1)^2} + \frac{1}{\sin^2\chi}\frac{\partial^2}{\partial(\varphi^2)^2}, \tag{4.11}$$

$$\triangle(S_5) = \frac{1}{\sin^3\gamma\cos\gamma}\frac{\partial}{\partial\gamma}\left[\sin^3\gamma\cos\gamma\frac{\partial}{\partial\gamma}\right] + \frac{1}{\cos^2\gamma}\frac{\partial^2}{\partial\varphi^{3^2}} + \frac{1}{\sin^2\gamma}\triangle(S_3)\,. \tag{4.12}$$

Next step is to find the wave function for our system. For this purpose we consider the Schrodinger equations which take for three-sphere the form:

$$\begin{aligned} \Delta_{S^3} &= \left[\frac{1}{\cos\chi\,\sin\chi}\frac{d}{d\chi}\left(\cos\chi\,\sin\chi\frac{d}{d\chi}\right) - \frac{m^2}{\cos^2\chi} - \frac{l^2}{\sin^2\chi} + J(J+2)\right] \\ \Delta_{S^3}&(\chi)U(\chi) = 0\,, \end{aligned} \tag{4.13}$$

and for five-sphere correspondingly

$$\begin{aligned} \Delta_{S^5} &= \left[\frac{1}{\cos\gamma\,\sin^3\gamma}\frac{d}{d\gamma}\left(\cos\gamma\,\sin^3\gamma\frac{d}{d\gamma}\right) - \frac{M^2}{\cos^2\gamma} - \frac{J(J+2)}{\sin^2\gamma} + L(L+4)\right] \\ \Delta_{S^5}&(\gamma)U(\gamma) = 0\,. \end{aligned} \tag{4.14}$$

The solutions to these equations are well known and they are:

$$U^J_{m,l}(\chi) = \tan^l\chi\,\cos^J\chi\,F\left[\frac{l-J+m}{2}, \frac{l-J-m}{2}; l+1; -\tan^2\chi\right], \tag{4.15}$$

$$U^L_{M,J}(\gamma) = \tan^J\gamma\,\cos^L\gamma\,F\left[\frac{J-L+M}{2}, \frac{J-L-M}{2}; J+2; -\tan^2\gamma\right]. \tag{4.16}$$

In addition we have to ensure that the solutions $U^J_{m,l}(\chi)$ and $U^L_{M,J}(\gamma)$ are square integrable with respect to the measures $d\mu(\chi) = \cos\chi\sin\chi\,d\chi$ and $d\mu(\gamma) = \cos\gamma\sin^3\gamma\,d\gamma$ with the following restrictions on the parameters

$$|m| + l = J - 2s\,, \qquad s = 0,1,2,\ldots,\left[\frac{J}{2}\right], \tag{4.17}$$

$$|M| + J = L - 2p\,, \qquad p = 0,1,2,\ldots,\left[\frac{L}{2}\right], \tag{4.18}$$

and $J > 0$, $L > 0$, $l > 0$. Introducing new variable $\sin^2\chi = \omega$, $k = \frac{1}{2}(|m| - m + 2s)$ (hence

$k \in \mathbb{N}$), we have:

$$\begin{aligned}
U^J_{m,l}(\chi) &= \tan^l\chi\,\cos^J\chi\,F\left[\frac{l-J+m}{2}\,,\,\frac{l-J-m}{2}\,;\,l+1\,;\,-\tan^2\chi\right]\\
&= \tan^l\chi\,\cos^J\chi\,F\left[\frac{m-|m|-2s}{2}\,,\,\frac{-m-|m|-2s}{2}\,;\,l+1\,;\,-\tan^2\chi\right]\\
&= \omega^{\frac{l}{2}}\,(1-\omega)^{\frac{|m|+2s}{2}}\;F\left[-k\,,\,-m-k\,;\,l+1\,;\,\frac{\omega}{\omega-1}\right]\\
&= \omega^{\frac{l}{2}}\,(1-\omega)^{\frac{|m|+2s}{2}}\;(1-\omega)^{-k}\;F\left[-k\,,\,m+k+l+1\,;\,l+1\,;\,\omega\right]\\
&= \omega^{\frac{l}{2}}\,(1-\omega)^{\frac{m}{2}}\;F\left[-k\,,\,m+k+l+1\,;\,l+1\,;\,\omega\right]\\
&= \frac{k!\,\Gamma(1+l)}{\Gamma(k+1+l)}\,\omega^{\frac{l}{2}}\,(1-\omega)^{\frac{m}{2}}\;P_k^{(l,m)}(1-2\omega)\,. \qquad (4.19)
\end{aligned}$$

In this notations the measure for $\chi \in S^3$ becomes

$$d\mu(\chi) = \cos\chi\sin\chi\,d\chi = \frac{1}{2}\,d\omega\,. \qquad (4.20)$$

One can easily find the normalization of our wave function computing

$$\begin{aligned}
\langle\, U^J_{m,l}\,|\,U^J_{m,l}\,\rangle &= \int_0^{\frac{\pi}{2}} {U^J_{m,l}}^2(\chi)\,d\mu(\chi)\\
&= \left[\frac{k!\,\Gamma(1+l)}{\Gamma(k+1+l)}\right]^2\frac{1}{2}\int_0^1 \omega^l\,(1-\omega)^m\,\left[P_k^{(l,m)}(1-2\omega)\right]^2\,d\omega\\
&= \frac{1}{2^{l+m+2}}\,\left[\frac{k!\,\Gamma(1+l)}{\Gamma(k+1+l)}\right]^2\int_{-1}^1 (1-x)^l\,(1+x)^m\,\left[P_k^{(l,m)}(x)\right]^2\,dx\\
&= \frac{1}{2(2k+l+m+1)}\,\left[\frac{k!\,\Gamma(1+l)}{\Gamma(k+1+l)}\right]^2\frac{\Gamma(k+l+1)\,\Gamma(k+m+1)}{k!\,\Gamma(k+l+m+1)}\,. \qquad (4.21)
\end{aligned}$$

Therefore, the normalized wave-function of (4.15) becomes:

$$\Psi^J_{m,l}(\omega) = \sqrt{\frac{2(2k+l+m+1)\,k!\,\Gamma(k+l+m+1)}{\Gamma(k+l+1)\,\Gamma(k+m+1)}}\;\omega^{\frac{l}{2}}\,(1-\omega)^{\frac{m}{2}}\;P_k^{(l,m)}(1-2\omega)\,. \qquad (4.22)$$

If we introduce $\sin^2\gamma = z$, $\alpha = J+1$, $\beta = M$, $n = \frac{1}{2}\,(|M|-m+2p)$ ($n \in \mathbb{N}$), the measure for the rest of gemetry becomes

$$d\mu(\gamma) = \cos\gamma\sin^3\gamma\,d\gamma = \frac{1}{2}\,z\,dz$$

and the normalized wave-function of (4.16) takes the form

$$\Psi^L_{M,J}(z) = \sqrt{\frac{2(2n+\alpha+\beta+1)\,n!\,\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}}\;z^{\frac{\alpha-1}{2}}\,(1-z)^{\frac{\beta}{2}}\;P_n^{(\alpha,\beta)}(1-2z). \qquad (4.24)$$

The potential should be also rewritten in terms of the new variables. We find for it

$$V(\gamma,\chi) = V(\omega,z) = \lambda\left[(n^3)^2 + z[(n^1)^2 - (n^3)^2 + \omega((n^2)^2 - (n^3)^2)]\right]. \tag{4.25}$$

Now we can compute the corrections to the energy using the perturbation theory. The final result for the first order correction to E^2 is

$$\begin{aligned} E^2_{(1)} &= \lambda \int_0^1 d\mu(\omega) \int_0^1 d\mu(z)\, V(\omega,z)\, {\Psi^{J}}^2_{m,l}(\omega)\, {\Psi^{L}}^2_{M,J}(z) \\ &= \lambda\Bigg\{(n^3)^2 + \frac{1}{2n+\alpha+\beta+1}\left[\frac{(n+\alpha)(n+\alpha+\beta)}{2n+\alpha+\beta} + \frac{(n+1)(n+\beta+1)}{2n+\alpha+\beta+2}\right] \times \\ &\qquad \times \Bigg[(n^1)^2 - (n^3)^2 + \left((n^2)^2 - (n^3)^2\right)\frac{1}{2k+l+m+1} \times \\ &\qquad\quad \times \left[\frac{(k+l)(k+l+m)}{2k+l+m} + \frac{(k+1)(k+m+1)}{2k+l+m+2}\right]\Bigg]\Bigg\}. \end{aligned} \tag{4.26}$$

A simple comparison of our result with that in [3] shows that the results differ significantly. This is because the string ansazt for the classical solutions in our case is different and therefore the solutions are actually new and different. The semicassical quantization we performed means summation over the classical solutions and therefore the final results should be different as actually is.

5. Conclusion

In this section we summarize the results of our study. The goal we pursued in this paper was to look for more general pulsating string solutions in $AdS_5 \times S^5$ background. Using the idea for generalized string ansatz suggested in [1] for folded string, we find new pulsating string solutions. In this case the string sigma model can be written as constrained system with the help of Lagrange multiplier Λ. We find two classes of solutions - for constant Λ and for non-constant Λ. While in the first case we obtained fairly general solution, in the latter one we restricted ourself to the case of $\alpha_3 = v_3 = r_3 = 0$. The corresponding solutions are found to be simple harmonic functions of τ ($\Lambda = const$) and Jacobi elliptic functions (non-constant Λ). Next we considered the corrections to the classical energy. From AdS/CFT point of view the corrections to the classical energy gives the anomalous dimensions of the operators in SYM theory and therefore they are of primary interest. Since we found solutions for particular ansatz for the calssical string solutions, to obtain the energy corrections we used the more general approach suggested by Minahan [2]. For this purpose we consider the Nambu-Goto actions and find the hamiltonian. After that we quantize the resulting theory semiclassically and obtain the corrections to the energy. Since we consider highly excited system, the kinetic term is dominating. All this means that we effectively perform summation over all classical solutions (not only those we explicitly found) while the effective potential term serves for a small perturbation. Since our ansatz, and therefore the solutions also, differs from that in [3], the final result for the corrections to the classical energy differs significantly.

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As a final comment we note that to comlete the analysis from AdS/CFT point of view, it is of great interest to develop Bethe ansatz analysis and to compare our result to that in SYM side. We leave this important question for future research.

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